Lecture 4. **Algebra of structural transformations**

We can obtain mathematical formulation of a dynamic system with the help of the following strategy:

1) First of all choose known junctions or fragments which have one input and one output.

2) Then, find all transfer functions $Wi(s)$ for all fragments ($\forall i = \overline{1,n}$).

3) Find transfer function $W(s)$ of the whole system.

4) Find $Xout(s)$ using equality $\{Xout(s)\} = \{W(s)Xin(s)\}$.

5) Obtain mathematical formulation in the time domain of the view $Xout(t) = f(Xin(t), K, T)$ using the inverse Laplace transformation $L^{-1}\{Xout(s)\} = L^{-1}\{W(s)Xin(s)\}$.

4.1 And now let us consider each point in more details e.g. we will carry out decomposition of dynamic system (fig. 2.2a) on separate fragments.
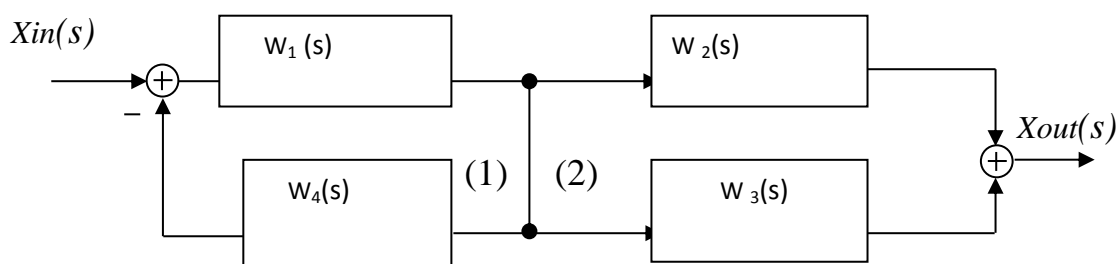

Fig. 2.2a. Fragment of a dynamic system structure

1) Definition of *the transfer function of a closed loop*.
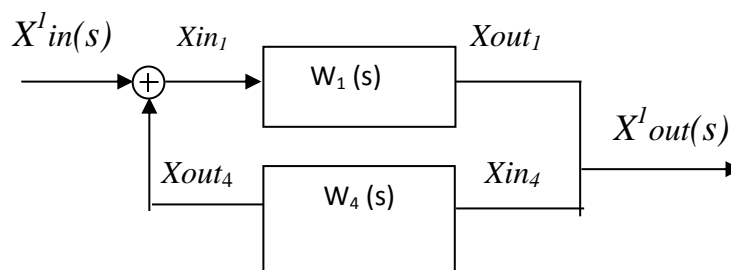Consider a fragment of the structure shown in fig. 2.2b.

Fig. 2.2b. A fragment of the structure of a dynamic system

a) Transfer functions: $W_1(s) \overset{\Delta}{=} \dfrac{X_{out_1}(s)}{X_{in_1}(s)}$; $W_4(s) \overset{\Delta}{=} \dfrac{X_{out_4}(s)}{X_{in_4}(s)}$.

b) Link equations: $X_{out_1}(s) = X_{in_4}(s) = X_{out}^1(s)$; $X_{in_1}(s) = X_{in}^1(s) + X_{out_4}(s)$.

Hence, $X_{in}^1(s) = X_{in_1}(s) - X_{out_4}(s)$.

1

c) The general transfer functions of the first fragment:

$$W^1(s) = \frac{X_{out_1}^1(s)}{X_{in}^1(s)} = \frac{X_{out_1}(s)}{X_{in_1} \mp X_{out_4}(s)}.$$

Divide numerator and denominator by $Xin(s)$ and then multiply one of the additives by $\dfrac{X_{out_1}}{X_{in_4}} \equiv 1$

i.e.
$$\frac{\dfrac{X_{out_1}(s)}{X_{in_1}(s)}}{1 \mp \dfrac{X_{out_4}(s)}{X_{in_1}(s)} * \dfrac{X_{out_1}(s)}{X_{in_4}(s)}} = \frac{W_1(s)}{1 \mp W_1(s)W_4(s)} = W^1(s).$$

The rule: Transfer function of *a closed loop system* is equal to the straight part transfer function divided by 1 plus/minus product of the straight part transfer function and feedback transfer function:

$$W_{closed}(s) = \frac{W_{dir}(s)}{1 \pm W_{dir}(s)W_{fb}(s)} . \qquad (2.24)$$
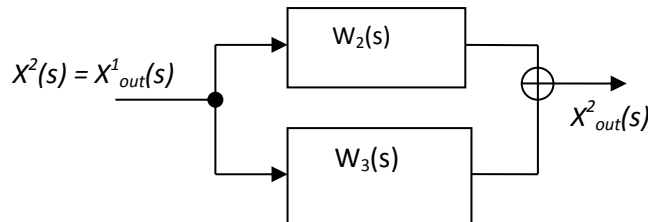
2) Consider the next fragment (fig. 2.2c):

Fig. 2.2c. A fragment of a dynamic system structure

The system is linear, so superposition principle is apply able.
The rule: the transfer function of links connected *in parallel* is equal to the sum of each link transfer function:

$$W(s) = \sum_{i=1}^{n} W_i(s) . \qquad (2.25)$$

In our case (fig. 2.2c): $W^2(s) = W_2(s) + W_3(s)$.

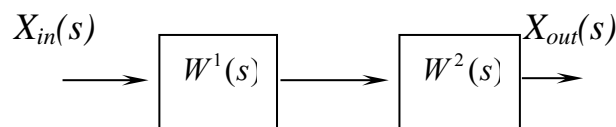3) The next fragment is *a series connection of several links.*

Fig. 2.2d. A fragment of a dynamic system structure

The rule: the transfer function of links connected in series is equal to the product of each link transfer functions:
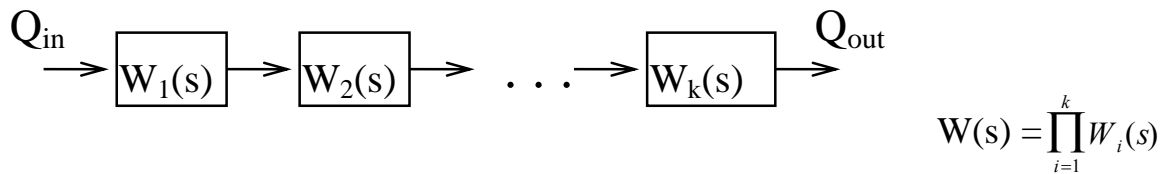
$$W(s) = \prod_{j=1}^{m} W_j(s). \qquad (2.26)$$

In our case (fig. 2.2d): $W(s) = W^1(s)W^2(s)$.

So, we have the transfer function of the whole system (fragment fig. 2.2a) as the following:
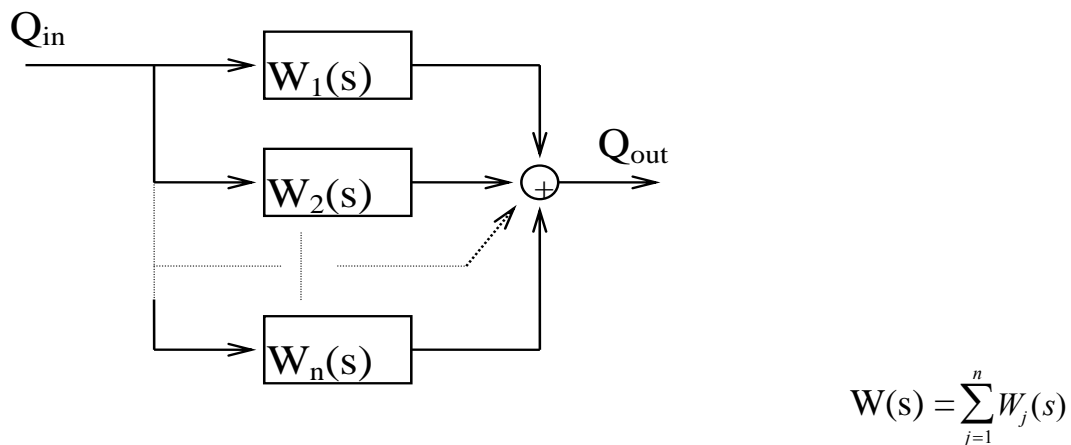
$$W(s) = \frac{W_1(s)[W_2(s) + W_3(s)]}{1 \pm W_1(s)W_4(s)}.$$

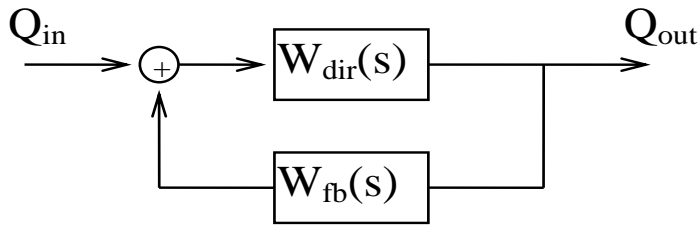### 4.2. *Main and additional transformation rules of skeleton diagrams*

1) Serial connection:

$$W(s) = \prod_{i=1}^{k} W_i(s)$$

2) Parallel connection:

$$W(s) = \sum_{j=1}^{n} W_j(s)$$

3) Closed loop:

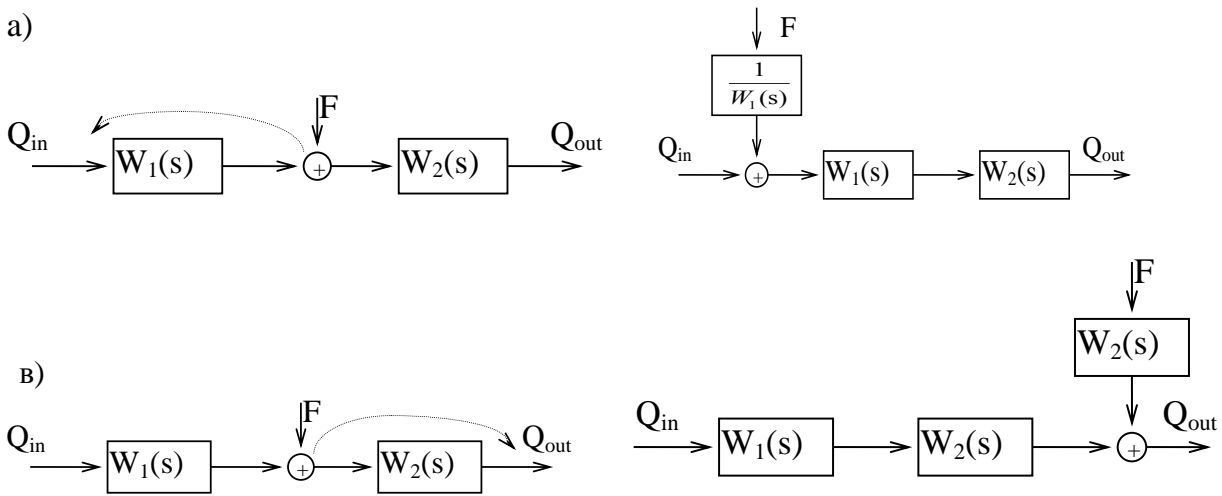$$Q_{in} \quad \boxed{+} \rightarrow \boxed{W_{dir}(s)} \quad Q_{out}$$
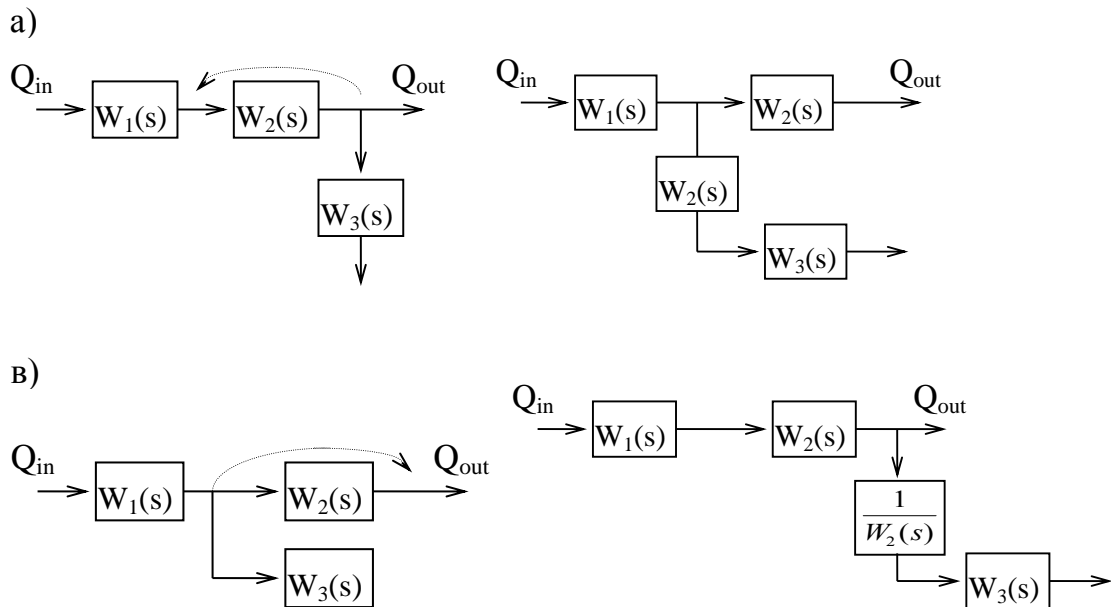
$$\boxed{W_{fb}(s)}$$

$$W_{closed}(s) = \frac{W_{dir}(s)}{1 \pm W_{dir}(s)W_{fb}(s)}$$

Note: The plus sign to be put if negative feedback; minus, if the positive back coupling.

4) Transfer of a node through a link:

а)

$Q_{in} \rightarrow \boxed{W_1(s)} \rightarrow \oplus \xleftarrow{F} \rightarrow \boxed{W_2(s)} \rightarrow Q_{out}$

$F \rightarrow \boxed{\dfrac{1}{W_1(s)}}$

$Q_{in} \rightarrow \oplus \rightarrow \boxed{W_1(s)} \rightarrow \boxed{W_2(s)} \rightarrow Q_{out}$

в)

$Q_{in} \rightarrow \boxed{W_1(s)} \rightarrow \oplus \xleftarrow{F} \rightarrow \boxed{W_2(s)} \rightarrow Q_{out}$

$F \rightarrow \boxed{W_2(s)}$

$Q_{in} \rightarrow \boxed{W_1(s)} \rightarrow \boxed{W_2(s)} \rightarrow \oplus \rightarrow Q_{out}$

5) Transfer of a point of branching:

а)

$Q_{in} \rightarrow \boxed{W_1(s)} \rightarrow \boxed{W_2(s)} \rightarrow Q_{out}$

$\boxed{W_3(s)}$

$Q_{in} \rightarrow \boxed{W_1(s)} \rightarrow \boxed{W_2(s)} \rightarrow Q_{out}$

$\boxed{W_2(s)}$

$\boxed{W_3(s)}$

в)

$Q_{in} \rightarrow \boxed{W_1(s)} \rightarrow \boxed{W_2(s)} \rightarrow Q_{out}$

$\boxed{W_3(s)}$

$Q_{in} \rightarrow \boxed{W_1(s)} \rightarrow \boxed{W_2(s)} \rightarrow Q_{out}$

$\boxed{\dfrac{1}{W_2(s)}}$

$\boxed{W_3(s)}$

The example of finding of transfer function of all system with use of the basic and additional rules of structural conversion is given below.

Example. The structural diagram of the following look is set:
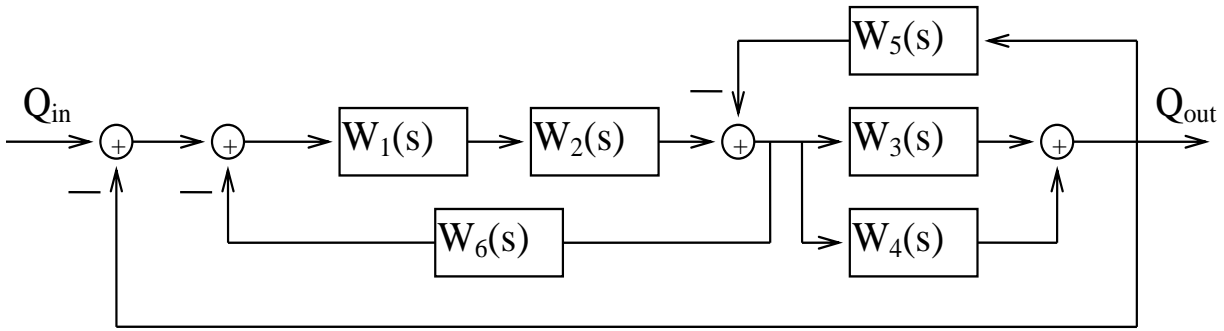
To receive transfer function of system if transfer functions of separate links are set $W_1(s) \div W_6(s)$.
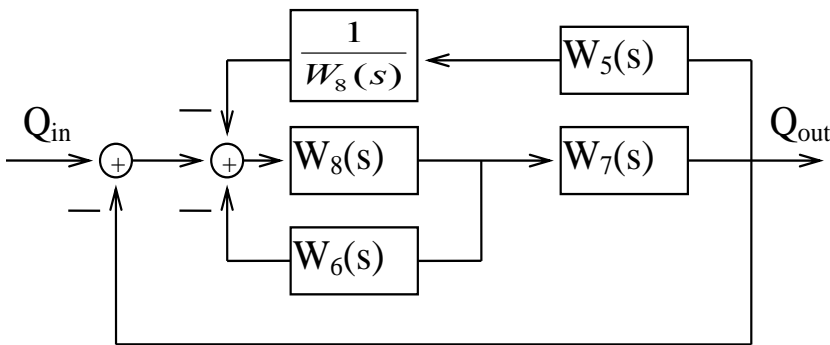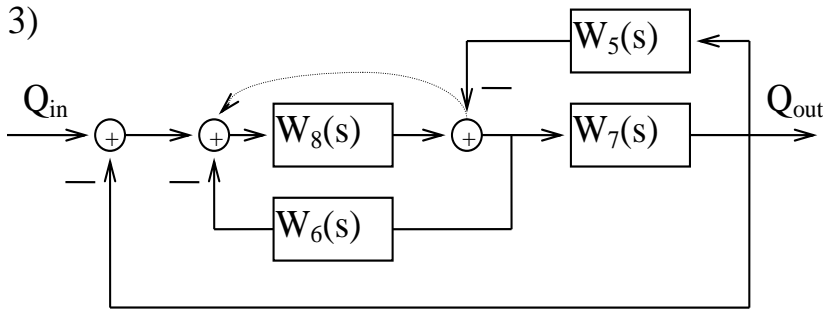
Algorithm and solving

1) We carry out decomposition of the given system on the known circuits; we write transfer functions of the selected circuits according to rules.

$$W_7(s) = W_3(s) + W_4(s)$$

2) $W_8(s) = W_1(s)\, W_2(s)$

We show what we do conversions:

3)

We receive transfer functions of the intermediate links:

$$W_9(s) = \frac{W_8(s)}{1 + W_6(s)W_8(s)} = \frac{W_1(s)W_2(s)}{1 + W_1(s)W_2(s)W_6(s)}$$

$$W_{10}(s) = W_7(s)\, W_9(s) = [\,W_3(s) + W_4(s)]\, W_9(s)$$

$$W_{11}(s) = \frac{W_{10}(s)}{1 + \dfrac{W_5(s)W_{10}(s)}{W_8(s)}} = \frac{W_8(s)W_{10}(s)}{W_8(s) + W_5(s)W_{10}(s)}$$

5) We receive transfer function of the whole system:

$$W(s) = \frac{W_{11}(s)}{1 + W_{11}(s)} .$$

6) Having added the given transfer functions of separate links in transfer function of the whole system, we will receive required transfer function of system through basic data:

$$W(s) = \frac{W_1(s)W_2(s)[W_3(s) + W_4(s)]}{1 + W_1(s)W_2(s)W_6(s) + W_5(s)[W_3(s) + W_4(s)] + W_1(s)W_2(s)[W_3(s) + W_4(s)]} .$$

There exist many other rules of transformation, in addition to those mentioned so far, like junction or branch point transfer. But the most general technique states that first of all the three patterns should be found: parallel connections, serial connections, closed loops. Then appropriate transformation rules should be applied sequentially, to obtain the final transfer function of the whole system. *Note that elements of the block diagram theory are identical to the graph theory elements.*

We have just solved the problem of composing the dynamical system from its elements. But what about the opposite task – decomposing a given system into structural blocks? It is clear that for the task we need to somehow classify internal elements according to some feature. It is customary to choose as the feature proper numbers of the system characteristic equation (or matrix A), since they determine the system properties. We will distinguish four types of them:

1) Zero roots: $Q_2(s) = Ts = 0$; $s \equiv 0$.

2) Real roots: $Q_2(s) = Ts + 1 = 0$; $s = -1/T$.

3) Complex conjugate roots: $Q_2(s) = T^2 s^2 + 2\xi Ts + 1 = 0$; $Q < \xi < 1$; $s_{1,2} = -\alpha \pm j\beta$.

4) Imaginary roots: $Q_2(s) = T^2 s^2 + 1 = 0$; $s_{1,2} = \pm j\beta$.

According to these four types we will consider now four very basic dynamic links.